Math 10A
Worksheet, Discussion #10; Friday, 6/29/2018
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1 Related Rates

1. A circle's area is expanding at a constant rate of $5m^2/s$. How fast is its radius changing when its area is $100\pi m^2$?

> **Solution:** The area of a circle is given by $A = \pi r^2$. Taking the derivative, we have that $A' = 2\pi r r'$. Now we plug in the values that we are give. We know that the area is increasing at a constant rate of 5 so $A' = 5$ and when $A = 100\pi$, we know that $r = \sqrt{100} = 10$. So $5 = 2\pi(10)r'$ so $r' = \frac{5}{20\pi} = \frac{1}{4\pi} m/s$.

2. A conical cup that is $6cm$ wide at the top and $5cm$ tall is filled with water is punctured at the bottom and water is coming out at a rate of $10^{-6}m^3/s$. Initially, the cup is filled How fast is the height of the water changing when the height is $2cm$?

> **Solution:** If the height of the water is h, then the radius of the cone formed by the water would be $3/5h$ and so the volume of the water cone is $V = \pi/3(3/5h)^2 \cdot h = \dfrac{3\pi h^3}{25}$. Taking the derivative of both sides gives
>
> $$V' = \frac{9\pi h^2 h'}{25}$$
>
> and plugging in -10^{-6} for V' and $2 \cdot 10^{-2}$ for h gives
>
> $$-10^{-6} = \frac{9\pi 4 \cdot 10^{-4} h'}{25} \implies h' = \frac{-1}{144\pi} m/s.$$

3. A lamppost is $5m$ tall. A woman who is $2m$ tall is walking away from it at a constant rate of $10cm/s$. When she is $2m$ away from the lamppost, how fast is her shadow length changing?

> **Solution:** Using similar triangles, if the woman is at a distance d from the lamppost and the shadow height is h, then
>
> $$\frac{h}{2} = \frac{h+d}{5} \implies 2d = 3h.$$

Taking the derivative, we have that $2d' = 3h'$ and $d' = 10cm/s$ so $h' = \frac{20}{3}cm/s$.

4. Sand is being dumped in a conical pile whose width and height always remain the same. If the sand is being dumped in at a rate of $2m^3/hr$, how fast is the height of the sand changing when the pile is $10cm$ tall?

Solution: Let the height of the pile be h. Then the radius of the pile is $r = \frac{h}{2}$ and the volume of the pile is $V = \frac{\pi r^2 h}{3} = \frac{\pi h^3}{12}$. Taking the derivative gives $V' = \frac{\pi}{4}h^2 h'$. Now we plug in 2 for V' and 10^{-1} for h to get $h' = \frac{800}{\pi}m/hr = \frac{800}{3600\pi}m/s = \frac{2}{9\pi}m/s$.

5. A ladder $5m$ tall is lying against a wall. The bottom of the ladder is pulled out at a rate of $10cm/s$. How fast is the area of the triangle formed by the ladder, wall, and floor changing when the bottom of the ladder is $3m$ away from the wall?

Solution: Let d be how far the bottom of the ladder is away from wall. Then the area of the triangle formed is $\frac{1}{2} \cdot d \cdot \sqrt{25 - d^2} = A$. Squaring both sides gives $4A^2 = d^2(25 - d^2)$. Now we can take the derivative to get that $8AA' = 2dd'(25 - d^2) + d^2(-2dd')$. When $d = 3$, the area is $\frac{1}{2} \cdot 3 \cdot 4 = 6$ and so

$$8 \cdot 6 \cdot A' = 2 \cdot 3 \cdot d'(16) + 9(-6d') \implies 48A' = 42d'.$$

Since $d' = 10^{-1}m/s$, we have that $A' = \frac{7}{80}m/s$.

6. A conical volcano is $100m$ tall and the base has a radius of $50m$. It is filling with lava at a rate of $\pi m^3/s$. At what rate is the height of the lava rising with it is $50m$ tall?

Solution: Let h be the height of the lava. The we can calculate the volume of the truncated cone by taking the total area and subtracting the missing top cone. The top cone has a height of $100 - h$ and radius of $(100 - h)/2$. Thus the volume of the lava is

$$V(h) = \frac{\pi \cdot 50^2 \cdot 100}{3} - \frac{\pi \cdot (100 - h)^2 \cdot (100 - h)}{2^2 \cdot 3}.$$

Taking the derivative, we get that

$$\frac{dV}{dt} = -\frac{\pi(100 - h)^2(-h')}{4}.$$

Since $V' = \pi$, we have that $h' = \frac{4}{50^2} = \frac{1}{625}$.

2 L'Hopital's Rule

7. Find $\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{3x}$.

> **Solution:** We use the trick of turning exponents into products by taking e to the ln of the function. So doing this gives
>
> $$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{3x}=\lim_{x\to\infty}\exp\left[\ln\left(1+\frac{1}{2x}\right)^{3x}\right]=\exp\left[\lim_{x\to\infty}3x\ln\left(1+\frac{1}{2x}\right)\right].$$
>
> Plugging in ∞ gives $\infty\cdot 0$ which is a product indeterminate and so we can turn this product into a quotient. Doing so gives
>
> $$\lim_{x\to\infty}3x\ln\left(1+\frac{1}{2x}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{2x}\right)}{(3x)^{-1}}=\lim_{x\to\infty}\frac{\frac{1}{1+1/2x}(-2(2x)^{-2})}{-3(3x)^{-2}}$$
>
> $$=\lim_{x\to\infty}\frac{3}{2+\frac{1}{x}}=\frac{3}{2}.$$
>
> Thus the answer to the original limit is $e^{3/2}$.

8. Find $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$.

> **Solution:** Plugging in $x=4$ gives $0/0$ which is indeterminate. Now we use LHopitals rule to get
>
> $$\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{1}{1/(2\sqrt{x})}=\lim_{x\to 4}2\sqrt{x}=4.$$

9. Find $\lim_{x\to 0}\frac{x\tan x}{\sin 3x}$.

> **Solution:** Plugging in $x=0$ gives $0/0$ so using Lhopitals gives
>
> $$\lim_{x\to 0}\frac{x\tan x}{\sin 3x}=\lim_{x\to 0}\frac{x\sec^2(x)+\tan x}{3\cos(3x)}=\frac{0}{3}=0.$$

10. Find $\lim_{x\to 0}\frac{\sin(x^2)}{x\tan x}$.

Solution: Plugging in 0 gives 0/0 and so we can use LHopitals rule to get

$$\lim_{x\to 0}\frac{\sin(x^2)}{x\tan x} = \lim_{x\to 0}\frac{2x\cos(x^2)}{x\sec^2(x)+\tan x}.$$

Plugging in 0 again gives 0/0 yet again, so we use LHopital's again to get

$$\lim_{x\to 0}\frac{2x\cos(x^2)}{x\sec^2(x)+\tan x} = \lim_{x\to 0}\frac{2\cos x^2 - 4x^2\sin x^2}{2\sec^2(x)+2x\tan x\sec^2(x)} = \frac{2-0}{2+0} = 1.$$

11. Find $\lim_{x\to 0}\frac{x^2e^x}{\tan^2 x}$.

Solution: Plugging in 0 gives 0/0 so we use LHopitals to get

$$\lim_{x\to 0}\frac{x^2e^x}{\tan^2 x} = \lim_{x\to 0}\frac{2xe^x + x^2e^x}{2\tan x\sec^2 x}.$$

Plugging in 0 again gives 0/0 so we use LHopitals again to get

$$\lim_{x\to 0}\frac{2xe^x + x^2e^x}{2\tan x\sec^2 x} = \lim_{x\to 0}\frac{x^2e^x + 4xe^x + 2e^x}{2\tan x(2\sec x\cdot\sec x\tan x) + 2\sec^4 x} = \frac{2}{2} = 1.$$

12. Find $\lim_{x\to\infty}(\sqrt{x^2+1} - \sqrt{x+1})$.

Solution: Plugging in ∞ gives $\infty - \infty$ which is an indeterminate. This is not a quotient so we can't use LHopital's yet. But we can try to multiply by the conjugate to get

$$\lim_{x\to\infty}(\sqrt{x^2+1} - \sqrt{x+1}) = \lim_{x\to\infty}\frac{x^2+1-(x+1)}{\sqrt{x^2+1}+\sqrt{x+1}}.$$

Now we plug in ∞ to get ∞/∞ so we can use LHopitals and get

$$= \lim_{x\to\infty}\frac{2x-1}{x/\sqrt{x^2+1} + 1/(2\sqrt{x+1})} = \lim_{x\to\infty}\frac{2x-1}{1/\sqrt{1+1/x^2} + 1/(2\sqrt{x+1})} = \frac{\infty}{1+0} = \infty.$$

Note that we could have solved it after multiplying by the conjugate by dividing the top and bottom by the largest power of x we saw, which was x^2. Doign so gives

$$\text{dim}_{x\to\infty}\frac{x^2-x}{\sqrt{x^2+1}+\sqrt{x+1}} = \lim_{x\to\infty}\frac{1-1/x}{\sqrt{1/x^2+1/x^4}+\sqrt{1/x^3+1/x^4}} = \infty/0 = \infty.$$

13. Find $\lim_{x\to 0^+} \ln x \cdot \tan x$.

Solution: Plugging in 0 gives $(-\infty)\cdot 0$. So, we can write it as

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{1/x}{-\csc^2(x)} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x}.$$

Plugging in 0 gives 0/0 so we can use LHopitals again to get

$$= \lim_{x\to 0^+} \frac{-2\sin x\cos x}{1} = 0.$$

14. Find $\lim_{x\to 0^+} x^{\sin x}$.

Solution: We don't like having x raised to some function of x so we do our trick of taking e to the ln of the function. This gives

$$\lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} \exp(\ln x^{\sin x}) = \exp\left[\lim_{x\to 0^+} \sin x \ln x\right].$$

Calculating the inner limit gives

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc(x)} = \lim_{x\to 0^+} \frac{1/x}{-\cot(x)\csc(x)} = \lim_{x\to 0^+} \frac{-\sin^2(x)}{\cos(x)x}.$$

Plugging in 0 again gives 0/0 so we use LHopitals again to get

$$\lim_{x\to 0^+} \frac{-2\sin x\cos x}{\cos x - x\sin x} = 0.$$

So our original answer is $e^0 = 1$.